Farkas’ lemma

Lemma
Consider $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then

1. $\exists x \geq 0$ such that $Ax = b$, or
2. $\exists p$ such that $p^T A \geq 0^T$ and $p^T b < 0$,

but not both.

Therefore, either

1. a (non-negative) solution to a system of equations, or
2. a solution to a system of inequalities (one of which strict).

Proof (easy)
We need to prove that (1) ⇒ ¬(2) and ¬(1) ⇒ (2).

(1) ⇒ ¬(2): If $x \geq 0$, $Ax = b$, and $p^T A \geq 0^T$, then

$$p^T b = p^T A x \geq 0,$$

a contradiction.

¬(1) ⇒ (2): Consider the primal-dual pair

\[
\begin{align*}
\text{P} : \max & \quad 0^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0 \\
\text{D} : \min & \quad p^T b \\
\text{s.t.} & \quad p^T A \geq 0^T
\end{align*}
\]

¬(1) ⇔ (3) \Rightarrow P is infeasible.

$p = 0$ is feasible for $D$, which must then be unbounded

⇒ There exists a feasible $p$ such that $p^T b < 0$

More on Farkas’ lemma

Corollary 4.3
Consider $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If, for any $p \in \mathbb{R}^m$,

$$p^T A \geq 0 \Rightarrow p^T b \geq 0,$$

then there exists $x \geq 0$ such that $Ax = b$. 
Closed sets and polyhedra

**Definition:** A set $S \subseteq \mathbb{R}^n$ is **closed** if it contains the limit of any sequence of its elements:

$$I = (x_i^j)_{i=1}^\infty : \exists \lim_{i \to \infty} x_i \Rightarrow I \subseteq S$$

**Theorem**

Every polyhedron is closed.

**Proof:**

- Any half-space $a^\top x \leq b$ is closed:

$$a^\top \left( \lim_{i \to \infty} x_i \right) = \lim_{i \to \infty} a^\top x_i \leq b.$$  

- The intersection of closed sets is closed.

Weierstraß’ theorem

**Weierstraß’ theorem**

Given $f : \mathbb{R}^n \to \mathbb{R}$ continuous, $S \subset \mathbb{R}^n$ nonempty, bounded, and closed,  

- $\exists x^\star \in S : f(x^\star) \leq f(x) \ \forall x \in S$  
- $\exists y^\star \in S : f(y^\star) \geq f(y) \ \forall y \in S$

That is, $S$ closed implies min and max $x^\star$ and $y^\star$ are both in $S$.

Example of non-closed $S$:

$$f(x) = -x \quad S = \{x \in \mathbb{R} : 1 \leq x < 2\}.$$  

$S$ is not closed. There exists a $y^\star = 1$, but no $x^\star \in S$.

Separating hyperplane theorem

Closed, **convex** sets have an important feature:

**Theorem**

Given a convex, closed set $S \subset \mathbb{R}^n$ and $x^\star \in \mathbb{R}^n \setminus S$, there exists $c \in \mathbb{R}^n$ such that  

$$c^\top x^\star < c^\top x \quad \forall x \in S$$

**Proof sketch:**

- Define $y = \arg\min\{||x - x^\star||_2 : x \in S\}$  
- Note: $y$ exists by Weierstraß’ theorem, and is unique  
- Define $c = y - x^\star$

Proof of Farkas’ lemma (hard)

**Lemma**

Consider $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then

1. $\exists x \geq 0$ such that $Ax = b$, or
2. $\exists p$ such that $p^\top A \geq 0^\top$ and $p^\top b < 0$,

but not both.

**Proof:** (1) $\Rightarrow$ (2) already proved.

$\neg$(1) $\Rightarrow$ (2): Define

$$S = \{y \in \mathbb{R}^m : y = Ax, x \geq 0\} \neq b$$

- $S$ is non-empty (it contains 0),  
- $S$ is convex (it is a linear transformation of the 1st orthant),  
- $S$ is closed (it is a polyhedron)
Proof of Farkas’ lemma (hard, cont.)

**Hypothesis:** ¬(1) ⇒ (2), i.e., ∄x ≥ 0 such that Ax = b
We must prove that ∃p such that p ⊤ A ≥ 0 ⊤ and p ⊤ b < 0

Hp ⇒ b /∈ S and we can separate b from S with a hyperplane.
I.e., there exists p ∈ R m such that p ⊤ b < p ⊤ y for any y ∈ S.

First, y = 0 ∈ S means p ⊤ b < 0. Second, any vector λA i for λ > 0 and any column A i of A is in S, as λA i = A(λe i).

But by the separating hyperplane theorem,

\[ p \top b < p \top (\lambda A_i) \]

and since this holds for λ > 0 we must have p ⊤ A i ≥ 0, and

\[ p \top A \geq 0 \top \]

Classical proof of strong duality

Consider the primal-dual pair

\[
\begin{align*}
P & : \min c \top x \\
D & : \max u \top b \\
\text{s.t.} & \ Ax \geq b \\
\text{s.t.} & \ u \top A = c \top \\
u & \geq 0
\end{align*}
\]

and suppose x* is feasible for P.

- Active constraints (at x*) by index set I: ∀i ∈ I, a_i \top x^* = b_i
- If there is a feasible direction, i.e., ∃d ∈ R^n : a_i \top d ≥ 0 ∀i ∈ I,
  ⇒ Optimality of x^* implies c \top (x^* + ɛd) ≥ c \top x^*, or c \top d ≥ 0
- by Corollary 4.3 of FL: if there is d such that Ad ≥ 0 implies c \top d ≥ 0, then there must exist u ∈ R^m_+ such that u \top A = c \top
- u has positive components u_i for i ∈ I, hence

\[ u \top b = \sum_{i \in I} u_i b_i = \sum_{i \in I} u_i a_i \top x_i^* = c \top x^* \]